UNIT-III (15M)

MULTIPLE INTEGRAL

Double Integral :- Let $f(x,y)$ be a function of '2' variables 'x' & 'y' defined on a bounded region 'R' on xy-plane.

Let us divide the region 'R' into 'n'-sub regions, each of area is $\delta R_1, \delta R_2, ----, \delta R_n$.

Consider the sum $\sum_{i=1}^{n} f(x_i, y_i)\, \delta R_i$

as $n \to \infty$ & $\delta R_i \to 0$ if

$\lim_{\substack{n\to\infty \\ \delta R_i \to 0}} f(x_i, y_i)\, \delta R_i$ exists then the limit value is called double integral of $f(x,y)$ in the region 'R'.

It is denoted by $\iint_R f(x,y)\, dx\, dy$.

Evaluation of double integrals :-

case(i) :- If the Region 'R' is bounded by the lines $a \le x \le b$, $c \le y \le d$ then

$$\iint_R f(x,y)\, dx\, dy = \int_{x=c}^{b} \left[\int_{y=c}^{d} f(x,y)\, dy \right] dx$$

(or)

$$= \int_{y=c}^{d} \left[\int_{x=c}^{b} f(x,y)\, dx \right] dy$$

the above integral can be evaluated by integrating w.r.to 'x' treated as 'y' as constant & then integrate w.r.to 'y'.

(or)

Integrate w.r.to 'y' treated as 'x' as constant & then integrate w.r.to 'x'.

Case-II :- If the region 'R' is bounded by $a \le x \le b$, $f_1(x) \le y \le f_2(x)$

(or)

$y_1(x) \le y \le y_2(x)$

$$\therefore \iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy \right] dx$$

i.e., 1st integrate w.r.to 'y' treat 'x' as constant & then integrate w.r.to 'x'.

Case-III :- If the region 'R' is bounded by $c \le y \le d$ then $x_1(y) \le x \le x_2(y)$

(or)

$f_1(y) \le x \le f_2(y)$.

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d} \left[\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx \right] dy$$

Integrate w.r.to 'x' treat as 'y' as constant & then integrate w.r.to 'y'.

Evaluate the following double Integrals.

1). $\int_0^2\int_0^x y\,dy\,dx$

sol:- $\int_0^2\left(\frac{y^2}{2}\right)_0^x dx = \int_0^2\left(\frac{x^2}{2}\right)dx = \left(\frac{x^3}{6}\right)_0^2 = \frac{8}{6} = 4/3.$

②. $\int_0^2\int_0^x e^{x+y}\,dx\,dy.$

sol:- $= \int_0^2 e^x\left[\int_0^x e^y\,dy\right]dx$

$= \int_0^2 e^x\left(e^y\right)_0^x dx = \int_0^2 e^x\left(e^x-1\right)dx = \left(\frac{e^{2x}}{2}-e^x\right)_0^2 = \left(\frac{e^4}{2}-e^2\right)-\left(\cancel{1}\right)\left(\frac{1}{2}-1\right)$

$= \frac{e^4}{2}-e^2+1/2.$

3). $\int_{x=0}^{2}\int_{y=0}^{3} xy\,dx\,dy.$ (9)

④. Evaluate $\iint_R y\,dx\,dy$, where 'R' is the region bounded by the parabolas $y^2=4x$ & $x^2=4y$
L(1) L(2)

sol:- subn (1) in (2)

$(1) \Rightarrow x = \frac{y^2}{4}$ & $(2) \Rightarrow \left(\frac{y^2}{4}\right)^2 = 4y \Rightarrow \frac{y^4}{16} = 4y \Rightarrow y^4-64y=0$

$\Rightarrow y(y^3-64)=0$

$y=0$ & $y=4.$

put $y=0$ in (1) $\Rightarrow x=0$

$y=4$ in (1) $\Rightarrow x=4.$

$\therefore (x,y)=(0,0) = O$

$(x,y)=(4,4) = A$

here 'R' is the region which is bounded by given two parabolas and these 2-parabolas intersect at a point (0,0) & (4,4)

Let us drawn on a strip parallel to Y-axis and fixed 'x'. (4)

For fixed 'x' the limits are $x: 0 \to 4$

$y = \frac{x^2}{4} \to 2\sqrt{x}$

$$\iint_R y\,dx\,dy = \int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y\,dy\right] dx$$

$$= \int_0^4 \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left(\frac{4x^2}{2} - \frac{x^5}{5\times 16}\right)_0^4$$

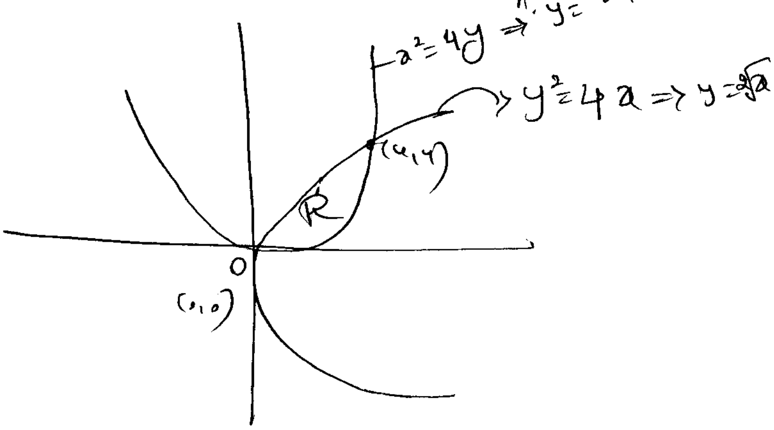
$$= \frac{1}{2}\left[2\times 16 - \frac{4\times 4\times \cancel{4}\times 4\times \cancel{4}}{\cancel{16}\times 5}\right]$$

$$= \frac{1}{2}\left[32 - \frac{64}{5}\right]$$

$$= \frac{48}{5}.$$

$||^{ly}$ fixed 'y' $\iint_R y\,dx\,dy = \int_0^4 \left[\int_{y^2/4}^{2\sqrt{y}} y \quad dx\right] dy$

$$= \frac{48}{5}$$



(5). $\int_{0}^{a}\int_{y=0}^{b}(x^2+y^2)\,dx\,dy$

$= \int_{x=0}^{a}\left[\int_0^b x^2 dy + \int_0^b y^2 dy\right] dx$

$= \int_0^a \left(x^2 y + \frac{y^3}{3}\right)_0^b dx = \int_0^a \left[\left(b\cancel{x}^2 + \frac{b^3}{3}\right) - (0+0)\right] dx = \int_0^a bx^2 dx + \int_0^a \frac{b^3}{3} dx$

$= \left(\frac{bx^3}{3}\right)_0^a + \frac{b^3}{3}(x)_0^a = \frac{ba^3}{3} + \frac{ab^3}{3} = \frac{ab}{3}(a^2+b^2)$ //

(6). $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

Sol:-

$= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2+y^2)\,dy\right] dx$

$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx = \int_0^1 \left(x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3}\right) - \left(x^2 x + \frac{x^3}{3}\right) dx$

$= \int_0^1 \left(x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3} - x^3 - \frac{x^3}{3}\right) dx$

$= \left(\frac{x^{5/2+1}}{5/2+1} + \frac{1}{3}\frac{x^{3/2+1}}{\frac{3}{2}+1} - \frac{x^4}{4} - \frac{x^4}{12}\right)_0^1$

$= \frac{x^{7/2}}{7/2} + \frac{1}{3}\frac{x^{5/2}}{5/2} - \frac{1}{4} - \frac{1}{12}$

$= \frac{2}{7} + \frac{2}{15} - 1/4 - 1/12$

$= 3/35$ //

(7). $\int_0^4 \int_{y^2/4}^{y} \frac{y}{(x^2+y^2)}\,dx\,dy$

Sol:- $= \int_0^4 \int_{y^2/4}^{y} \left(y/x^2+y^2\right) dx\,dy$

$$= \int_0^4 y \left[\int_{y^2/4}^{y} \frac{1}{y^2+x^2}\, dx \right] dy$$

$$= \int_0^4 y \left[\frac{1}{y} \tan^{-1}(x/y) \right]_{x=y^2/4}^{y} dy$$

$$= \int_0^4 \frac{y}{y} \left[\tan^{-1}(y/y) - \tan^{-1}\left(\frac{y^2}{4y}\right) \right] dy$$

$$= \int_0^4 \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4}\right) \right] dy$$

$$= \frac{\pi}{4}(y)_0^4 - \int_0^4 \underset{u}{\tan^{-1}\left(\frac{y}{4}\right)} \cdot \underset{v}{1}\, dy$$

$$= \pi - \left[\left(\tan^{-1}\frac{y}{4}\; y \right)_0^4 - \int_0^4 \left(\frac{1}{1+\frac{y^2}{16}} \frac{y}{4} \right) dy \right]$$

$$= \pi - \left[\left(4\,\frac{\pi}{4} - 0 \right) - \int_0^4 \left(\frac{16}{16+y^2} \right) \frac{y}{4}\, dy \right]$$

$$= \pi - \left[\pi - \frac{16}{4} \int_0^4 \frac{y}{16+y^2}\, dy \right]$$

$$= 4 \int_0^4 \frac{y}{16+y^2}\, dy$$

$$= \frac{4}{2} \int_0^4 \frac{2y}{16+y^2}\, dy$$

$$= 2 \left[\log(16+y^2) \right]_0^4$$

$$= 2 \left[\log 32 - \log 16 \right]$$

$$= 2 \left[\log \frac{32}{16} \right]$$

$$= 2 \log 2$$

(8). $\int_0^4\int_0^{x^2} e^{y/x}\,dy\,dx$

Sol:- $\int_0^4\left[\int_0^{x^2} e^{y/x}\,dy\right]dx = \int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx = \int_0^4 x\left(e^{x^2/x} - e^0\right)dx = \int_0^4 x(e^x-1)\,dx$

$=\int_0^4 xe^x\,dx - \int_0^4 x\,dx$

$= \left[(xe^x)_0^4 - \int_0^4 e^x\,dx\right] - \left(x^2/2\right)_0^4$

$=(4e^4-0)-(e^4-e^0)-(8)$

$=4e^4-e^4-8+1 = 3e^4-7$ //

H.W

(9). $\int_{x=0}^{3}\int_{y=1}^{2} xy(1+x+y)\,dy\,dx = \frac{123}{4}$

(10). $\int_{x=0}^{5}\int_{y=0}^{x^2} x(x^2+y^2)\,dx\,dy = \frac{29}{24}\left[5^6\right]$ //

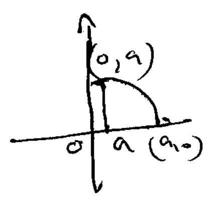
(12). $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$ $\left(\pi^2/4\right)$. (13). $\int_0^a\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$ $\left(\frac{\pi a^3}{6}\right)$

(14). Evaluate $\iint_R y^2\,dx\,dy$ where 'R' is the region bounded by the Parabolas $y^2=4x$, $x^2=4y$.

(15). Evaluate $\iint xy\,dx\,dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$

Sol:- $\iint xy\,dx\,dy = \int\left(\int x\,dx\right)y\,dy$ over 'R'.

$\therefore \iint xy\,dx\,dy = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{a^2-y^2}} x\,dx\right] y\,dy = \int_0^a \left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$



$=\frac{1}{2}\int_0^a (a^2-y^2)\,y\,dy = \frac{1}{2}\left(\frac{a^2y^2}{2}-\frac{y^4}{4}\right)_0^a$

$=\frac{1}{2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right) = \frac{1}{2}\left(\frac{2a^4}{8}\right) = a^4/8$ //

(16). Evaluate $\iint_R xy\,dx\,dy$, where 'R' is the region bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$.

Solⁿ: $x=2a$ & $x=2\sqrt{ay}$.

$$2a=2\sqrt{ay}$$

$$\Rightarrow a=\sqrt{ay} \Rightarrow a^2-ay=0$$

$$\Rightarrow a(a-y)=0$$

$$y=a.$$

$$\therefore \int_{y=0}^{a}\int_{x=2a}^{2\sqrt{ay}} xy\,dx\,dy = \frac{a^4}{3}.$$

(17). Find $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \Rightarrow \frac{x^2}{a^2}=1-\frac{y^2}{b^2} \quad (\text{or}) \quad \frac{y^2}{b^2}=1-\frac{x^2}{a^2}.$$

Solⁿ:

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$x=\pm a.$$

$$\therefore \int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy = \int_{-a}^{a}\left(x^2y+\frac{y^3}{3}+\frac{2xy^2}{2}\right)\Bigg|_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a}\left\{x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3}{3}+\frac{2x\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2}{2}\right\}dx - \int_{-a}^{a}\left\{x^2\left(-\frac{b}{a}\sqrt{a^2-x^2}\right)+\frac{\left(-\frac{b}{a}\sqrt{a^2-x^2}\right)^3}{3}+\frac{2x\left(-\frac{b}{a}\sqrt{a^2-x^2}\right)^2}{2}\right\}dx$$

$$= \int_{-a}^{a}\left\{\frac{bx^2}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\left(\sqrt{a^2-x^2}\right)^3+\cancel{\frac{xb^2}{a^2}(a^2-x^2)}+\frac{bx^2}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\left(\sqrt{a^2-x^2}\right)^3-\cancel{\frac{xb^2}{a^2}(a^2-x^2)}\right\}dx$$

$$= \int_{-a}^{a}\left\{2\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{2b^3}{3a^3}\left(\sqrt{a^2-x^2}\right)^3\right\}dx$$

$$= 2\cdot 2\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

$$= 4\int_{0}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

put $x=a\sin\theta \Rightarrow dx=a\cos\theta\,d\theta \Rightarrow x=0,\ \theta=0$ & $x=a,\ \theta=\pi/2$

$$\therefore I = 4\int_0^{\pi/2}\left[\frac{b}{a}a^2\sin^2\theta\, a\cos\theta + \frac{b^3}{3a^3}a^3\cos^3\theta\right]a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left[ba^3\sin^2\theta.\cos^2\theta\, d\theta + \frac{b^3a}{3}\cos^4\theta\, d\theta\right]$$

$$= 4\left[a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{b^3a}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta\right]$$

$$= 4\left[a^3b.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2} + \frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{4\pi}{2}\left[a^3b\frac{1}{8} + \frac{ab^3}{\cancel{3}}.\frac{\cancel{3}}{8}\right]$$

$$= \frac{\cancel{4}\pi}{\cancel{2}}\left[\frac{a^3b + ab^3}{8}\right]$$

$$= \frac{\pi}{4}(a^3b + ab^3)$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

$$\left[\because \int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta\right]$$

$$\left[\because \int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}\int_0^{\pi/2}(\sin 2\theta)^2 d\theta\right]$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)_0^{\pi/2}$$

$$\left[\because \cos 2\theta = 1-2\sin^2\theta\right.$$
$$\cos 4\theta = 1-2\sin^2 2\theta$$
$$\left.\sin^2 2\theta = \frac{1-\cos 4\theta}{2}\right]$$

$$= \frac{1}{8}\left(\frac{\pi}{2}-0\right) - \frac{1}{4}(0-0)$$

$$= \frac{1}{8}.\frac{\pi}{2}.$$

Ily

$$\int_0^{\pi/2}\cos^4\theta\, d\theta = \int_0^{\pi/2}(\cos^2\theta)^2 d\theta$$

$$= \int_0^{\pi/2}\left(\frac{\cos 2\theta + 1}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}(\cos^2 2\theta + 1 + 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left[\int_0^{\pi/2}\left(\frac{1+\cos 4\theta}{2}\right) + 1 + 2\cos 2\theta\right] d\theta$$

$$= \frac{1}{4}\left[\int_0^{\pi/2}\left(\frac{1+\cos 4\theta + 2 + 4\cos 2\theta}{2}\right)\right] d\theta$$

$$(\because \cos 2\theta = 2\cos^2\theta - 1) \quad = \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4} + 2\theta + \frac{4\sin 2\theta}{2}\right)$$

$$= \frac{1}{8}\left(\frac{\pi}{2} + \frac{2\pi}{2}\right) = \frac{1}{8}.\frac{3\pi}{2}$$

(18). Evaluate $\iint(x^2+y^2)\,dx\,dy$ in the +ve Quadrant for which $x+y \le 1$.

Sol:-
$x=0 \to x=1$
$y=0 \to y=1-x$. [Ans:- $1/6$]

(19). $\iint(x^2+y^2)\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\left[\text{Ans}: \frac{\pi ab}{4}(a^2+b^2)\right]$$

(19). Evaluate $\iint_R y\,dx\,dy$, where 'R' is the domain bounded by y-axis, the curve $y=x^2$ and the line $x+y=2$ in the 1st Quadrant.

$x=0 \to x=1$
$y=x^2 \to y=2-x$. ($16/15$)

(20). Evaluate $\iint x^2\,dx\,dy$ over the region bounded by hyperbola $xy=4$, $y=0$, $x=1$ & $x=4$ // (30)

Double Integrals in Polar co-ordinates :—

1. Evaluate $\int_0^{\pi}\int_a^{\sin\theta} r.\,dr.\,d\theta$

Sol:— $\int_0^{\pi}\left(\frac{r^2}{2}\right)_a^{\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi}\left(\sin^2\theta - a^2\right) d\theta = \frac{1}{2}\left[\int_0^{\pi}\sin^2\theta\, d\theta - \int_0^{\pi} a^2 d\theta\right]$

$= \frac{1}{2}\left[\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta - \left(a^2\theta\right)_0^{\pi}\right]$

$= \frac{1}{2}\left[\frac{\theta}{4} - \frac{\sin 2\theta}{4} - a^2\theta\right]_0^{\pi} = \frac{1}{2}\left(\frac{\pi}{4} - 0 - a^2\pi\right) - \frac{1}{2}(0) = \frac{\pi}{8} - \frac{a^2\pi}{2}$

2. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi}\left[\frac{1-\cos 2\theta}{2}\right] d\theta = \frac{a^2}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}$

$= \frac{a^2}{4}(\pi) = \frac{\pi a^2}{4}$

3. Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}.\,r\,d\theta\,dr$

Sol:— $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}.r\,d\theta\,dr = \int_0^{\infty} e^{-r^2}.r\,(\theta)_0^{\pi/2} dr = \int_0^{\infty} r e^{-r^2}\left(\frac{\pi}{2}\right) dr$

$= \frac{\pi}{2}\int_0^{\infty} r.e^{-r^2} dr$

$r^2 = t$; put $r=0$; $r=\infty$

$2r\,dr = dt$; $t=0$; $t=\infty$

$r\,dr = \frac{1}{2}dt$

$= \frac{\pi}{2}\int_{t=0}^{\infty}\frac{1}{2}e^{-t}dt$

$= \frac{\pi}{4}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} = -\frac{\pi}{4}\left(e^{-\infty} - e^0\right) = \pi/4$

4. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

Sol:— $\int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2 d\theta = \frac{a^2}{2}\int_0^{\pi}\left(1+\cos^2\theta + 2\cos\theta\right) d\theta$

$= \frac{a^2}{2}\left[\theta + 2\sin\theta + \frac{\theta + \frac{\sin 2\theta}{2}}{2}\right]_0^{\pi} = \frac{a^2}{2}\left[\pi + \frac{\pi}{2}\right] = \frac{3\pi a^2}{4}$

(5). Evaluate $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol: $$= \int_0^{\pi/4}\left\{\int_0^{a\sin\theta} \frac{r\,dr}{\sqrt{a^2-r^2}}\right\} d\theta = -\frac{1}{2}\int_0^{\pi/4}\left\{\int_0^{a\sin\theta} \frac{-2r}{\sqrt{a^2-r^2}}\,dr\right\} d\theta$$

$$\left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+c\right]$$

$$= -\frac{1}{2}\int_0^{\pi/4}\left[2\sqrt{a^2-r^2}\right]_{r=0}^{a\sin\theta} d\theta$$

$$= (-1)\int_0^{\pi/4}\left(a\cos\theta - a\right) d\theta$$

$$= (-a)\left(\sin\theta - \theta\right)_0^{\pi/4} = (-a)\left(\frac{1}{\sqrt2}-\frac{\pi}{4}\right) = a\left(\frac{\pi}{4}-1/\sqrt2\right)$$

(6). $\displaystyle\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$

Sol: $$= \int_{\theta=0}^{\pi/2}\left[\int_{a(1-\cos\theta)}^{a} r^2\,dr\right] d\theta = \int_0^{\pi/2}\left(\frac{r^3}{3}\right)_{a(1-\cos\theta)}^{a} d\theta = \frac{1}{3}\int_0^{\pi/2}\left[a^3 - a^3(1-\cos\theta)^3\right] d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left[1-(1-\cos\theta)^3\right] d\theta = \frac{a^3}{3}\int_0^{\pi/2}\left(\cancel{1}-\cancel{1}+\cos^3\theta+3\cos\theta-3\cos^2\theta\right) d\theta$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2}\cos^3\theta\,d\theta + 3\int_0^{\pi/2}\cos\theta\,d\theta - 3\int_0^{\pi/2}\cos^2\theta\,d\theta\right]$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2}\left(\frac{\cos3\theta+3\cos\theta}{4}\right) d\theta + 3\left(\sin\theta\right)_0^{\pi/2} - 3\left(\frac{\theta}{2}+\frac{\sin2\theta}{4}\right)_0^{\pi/2}\right] \quad \left[\because \cos3\theta = 4\cos^3\theta - 3\cos\theta\right]$$

$$= \frac{a^3}{3}\left[\left(\frac{\frac{\sin3\theta}{3}+3\sin\theta}{4}\right)_0^{\pi/2} + 3\left(\sin\theta\right)_0^{\pi/2} - \frac{3\pi}{2}\right]$$

$$= \frac{a^3}{3}\left[\frac{-\frac{1}{3}+3}{4} + 3 - 3\left(\frac{\frac{\pi}{2}+0}{2}\right)\right]$$

$$= \frac{a^3}{3}\left[\frac{-1+9}{3\cdot4} + 3 - 3\,\frac{\pi}{4}\right]$$

$$= \frac{a^3}{3}\left[2/3 + 3 - 3\pi/4\right] = \frac{a^3}{36}\left(44-9\pi\right)$$

7. Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

Sol:- $\int_0^{\pi/2}\int_0^{\infty} \frac{r.dr.d\theta}{(r^2+a^2)^2} = \int_0^{\pi/2}\left\{\int_0^{\infty} \frac{r\,dr}{(r^2+a^2)^2}\right\} d\theta$

$= \int_0^{\pi/2}\left[\int_{a^2}^{\infty} \frac{\frac{1}{2}\,dt}{t^2}\right\} d\theta$

$r^2+a^2 = t$ put $r=0 \Rightarrow t=a^2$

$2r\,dr = dt$ $r=\infty \Rightarrow t=\infty$

$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{-1}{t}\right)_{a^2}^{\infty} d\theta$

$= -\frac{1}{2}\int_0^{\pi/2} \left(\frac{-1}{\infty} + \frac{1}{a^2}\right) d\theta = \frac{1}{2}\int_0^{\pi/2}\left(0+\frac{1}{a^2}\right) d\theta = \frac{1}{2a^2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4a^2}$ //

8. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta \, dr\, d\theta$.

Sol:- Let $I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, dr\, d\theta$

$= \int_0^{\pi}\cos\theta\, d\theta \int_0^{a(1+\cos\theta)} r^2\, dr$.

$= \int_0^{\pi}\cos\theta\, d\theta \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)}$

$= \frac{a^3}{3}\int_0^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta$

$= \frac{a^3}{3}\int_0^{\pi} \cos\theta\left(1+\cos^3\theta + 3\cos\theta + 3\cos^2\theta\right) d\theta$

$= \frac{a^3}{3}\int_0^{\pi} \left(\cos\theta + \cos^4\theta + 3\cos^2\theta + 3\cos^3\theta\right) d\theta$

$\left(\because \cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1+\left(\frac{1+\cos 4\theta}{2}\right) + 2\cos 2\theta\right) = \frac{1}{4}\left[\frac{3+\cos 4\theta + 4\cos 2\theta}{2}\right]\right)$

$= \frac{a^3}{3}\left[\int_0^{\pi}\cos\theta + \frac{1}{4}\left(\frac{3+\cos 4\theta + 4\cos 2\theta}{2}\right) + 3\left(\frac{1+\cos 2\theta}{2}\right) + 3\left(\frac{\cos 3\theta + 3\cos\theta}{4}\right) d\theta\right]$

$= \frac{a^3}{3}\left[\sin\theta + \frac{1}{4}\left(\frac{3}{2}\theta + \frac{\sin 4\theta}{4\times 2} + \frac{4\sin 2\theta}{2\times 2}\right) + 3\left(\frac{\theta + \frac{\sin 2\theta}{2}}{2}\right) + \frac{3}{4}\left(\frac{\sin 3\theta}{3} + 3\sin\theta\right)\right]_0^{\pi}$

$= \frac{a^3}{3}\left[\left(0 + \frac{3\pi}{8} + \frac{3\pi}{2}\right) - 0\right] = \frac{a^3}{3}\left[0 + \frac{15\pi}{8}\right] = \frac{15\pi a^3}{8\times 3} = \frac{5\pi a^3}{8}$ //

(9). S.T $\iint r^2 \sin\theta \, dr\, d\theta = \frac{2a^3}{3}$, where 'R' is the Semi-circle $r = 2a\cos\theta$ above the initial line. $\left(\frac{2a^3}{3}\right)$

Sol:- $r = 2a\cos\theta$ & put $r = 0 \Rightarrow 0 = 2a\cos\theta$

$r = 0$

$\cos\theta = 0$

$\boxed{\theta = \pi/2}$

$\therefore \theta = 0 \to \pi/2$

(10). $\iint r^3 \, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$

Sol:- $2\sin\theta = 4\sin\theta \Rightarrow 4\sin\theta - 2\sin\theta = 0$

$2\sin\theta = 0$ $\Rightarrow 2\sin\theta = 0$

$\sin\theta = 0$ $\sin\theta = 0$

$\boxed{\theta = \pi}$. $\boxed{\theta = 0}$

$= \frac{45\pi}{2}$ //

(11). $\iint r\sin\theta \, dr\, d\theta$. over the cardioid $r = a(1+\cos\theta)$ above the initial line. $\left(\frac{4a^2}{3}\right)$ //.

$\theta = 0 \to \pi$.

$r = 0 \to a(1+\cos\theta)$

$r = 0 \Rightarrow 0 = a(1+\cos\theta)$

$\cos\theta = 1$

$\theta = 0, \pi$

Change of variables in double integral :-

In the given double (or) triple integral to make integration process is easy, we have to change variables in following methods.

①. change of variable from (x,y) to (u,v) :-

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(u,v)\,|J|\,du\,dv$$

Here $|J|$ = Jackobian of x,y w.r.t (u,v) &

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

②. change of cartesian co-ordinates to polar co-ordinates :-

Sub $x = r\cos\theta$
$y = r\sin\theta$

$$\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\,|J|\,dr\,d\theta$$

$$|J| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$; $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\qquad$ $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\qquad$ $\frac{\partial y}{\partial \theta} = r\cos\theta.$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r(1)$

$= r.$

$\therefore \iint f(x,y)\,dx\,dy = \iint f(r,\theta)\,r\,dr\,d\theta$

① Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar-co-ordinates. (15)

Hence S.T $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Sol The region of integration is given by

$x=0$ and $x=\infty$

$y=0$ and $y=\infty$.

Let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\boxed{x^2+y^2=r^2}$

Now $y=0 \Rightarrow r\sin\theta=0 \Rightarrow \boxed{\theta=0}$

$x=0 \Rightarrow r\cos\theta=0 \Rightarrow \boxed{\theta=\pi/2}$

$\theta: 0 \to \pi/2$.

i.e. The given region is a quadrant of the circle $x^2+y^2=r^2$.

In the region of the integration $r: 0 \to \infty$.

$$\therefore \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\right]d\theta = \int_{\theta=0}^{\pi/2}\left[\int_{t=0}^{t=\infty} e^{-t}\,\frac{dt}{2}\right]d\theta$$

$r^2=t$

$2r\,dr=dt$

$r\,dr=\frac{dt}{2}$

Put $r=0 \Rightarrow t=0$

$r=\infty \Rightarrow t=\infty$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}\left(e^{-t}\right)_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(0-1)\,d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2}$$

$$I = \frac{\pi}{4} \quad \text{--- (1).} \quad (2)$$

and also $I = \int_0^{\infty} e^{-x^2}dx \times \int_0^{\infty} e^{-y^2}dy = \left[\int_0^{\infty} e^{-x^2}dx\right]^2 = \frac{\sqrt{\pi}}{2}$ // [(1) + (2)]

(16)

(2). Evaluate the following integral by transforming into polar coordinates. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx\, dy$. $\left(\frac{a^4}{4}\right)$

Sol:- The region of integration is given by

$y=0$, $y=\sqrt{a^2-x^2}$ & $x=0$, $x=a$.

$y=0$, $y^2=a^2-x^2$

$x^2+y^2=a^2$

$\therefore$ the given region is a quadrant of the circle $x^2+y^2=a^2$.

$x=r\cos\theta$, $y=r\sin\theta$ & $dx\,dy = r\,dr\,d\theta$ & $x^2+y^2=r^2$

put $x=0 \Rightarrow \theta=0$; $x=a \Rightarrow r=a$.

$y=0 \Rightarrow \theta=\pi/2$; $\left[y=\sqrt{a^2-x^2} \Rightarrow a^2-x^2=r^2 \Rightarrow \cancel{x^2}+y^2-\cancel{x^2}=a^2\right.$
$\left. y^2=a^2\right]$ ✗

$r: 0 \to a$.

$\left[y=\sqrt{a^2-x^2} \Rightarrow x^2+y^2=a^2 \Rightarrow 4r^2=a^2 \Rightarrow 3r^2=0.\right.$
$\left. r=0\right]$ ✗

$\therefore \int_0^a\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} (r\sin\theta)\, r \cdot r\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^a r^3\sin\theta\, dr\, d\theta = \left(\frac{r^4}{4}\right)_0^a \int_0^{\pi/2} \sin\theta\, d\theta \Rightarrow \frac{a^4}{4}\left(-\cos\theta\right)_0^{\pi/2} = \frac{a^4}{4}$ //

(3). Evaluate the double integral $\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy\, dx$ $\left(\frac{\pi a^4}{8}\right)$

Sol:- $x=0$, $x=\sqrt{a^2-y^2}$ & $y=0$, $y=a$.

$x=0$, $x^2+y^2=a^2$

$x=r\cos\theta$
$y=r\sin\theta$ & $x^2+y^2=r^2$.

$dx\,dy = r\,dr\,d\theta$.

$\therefore$ $\theta=0 \to \pi/2$

$r=0 \to a$.

$\therefore \iint (x^2+y^2)\,dy\,dx = \frac{\pi a^4}{8}$ //

(4). Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing into Polar Co-ordinates.

Sol:- $x=0,\ x=2$ & $y=0,\ y=\sqrt{2x-x^2}$

$x=0,\ x=2$ & $y=0,\ y^2=2x-x^2$

$x^2+y^2=2x.$

put $x=r\cos\theta$; $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$ & $x^2+y^2=r^2$

put $x=0 \Rightarrow \theta=0$; $x=2 \Rightarrow x=2 \Rightarrow 2\cos\theta=2 \Rightarrow \boxed{\theta=0}$

$y=0 \Rightarrow \theta=\pi/2$; $x^2+y^2=2x \Rightarrow r^2=2r\cos\theta = r=2\cos\theta.$

$\therefore\ r: 0 \to 2\cos\theta.$; $\theta=0 \to \pi/2.$

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dx\, dy = \frac{3\pi}{4} \text{ //}$$

(5). S.T $\displaystyle\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$

(or)

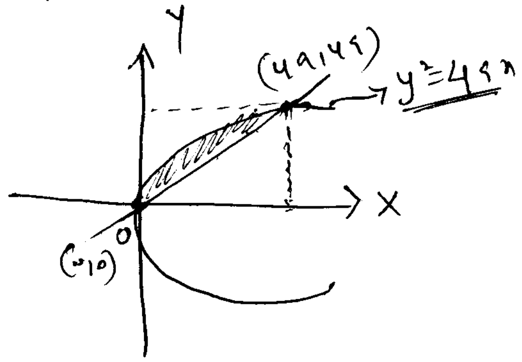
$\displaystyle\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy$ by changing to polar coordinates.

Sol:- The region of integration is given by

$x=\dfrac{y^2}{4a},\ x=y$ & $y=0,\ y=4a$

$y^2=4ax$; $y=x.$

i.e, the region is bounded by parabola $y^2=4ax$. and the straight line $y=x$.



Let $x=r\cos\theta$

$y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta.$

$x^2+y^2=r^2.$

From $y^2=4ax \Rightarrow r^{\cancel{2}}\sin^2\theta = 4a\cancel{r}\cos\theta \Rightarrow \boxed{r=\dfrac{4a\cos\theta}{\sin^2\theta}}$ $\therefore\ r: 0 \to \dfrac{4a\cos\theta}{\sin^2\theta}.$

From $y=x$ line with slope $m=1$

$\tan\theta=1$

$\theta=\pi/4$.

$\rightarrow \theta: \frac{\pi}{4} \rightarrow \pi/2$

$$\therefore \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta-\sin^2\theta)\,r\,dr\,d\theta.$$

$$=\int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta-\sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$[A: 8a^2(\frac{\pi}{2}-\frac{5}{3})]$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cot^4\theta-\cot^2\theta)\,d\theta$$

Evaluate ↓

$[\because \int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx]$

$$= 8a^2\left(\pi/2-\frac{5}{3}\right)$$

⑥. By changing into polar coordinates,

$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\,dy\,dx.$$

Sol:-

$y=x$, $y=\sqrt{2-x^2}$

$x^2+y^2=(\sqrt{2})^2$

$r: 0 \rightarrow \sqrt{2}$.

& $y=x$.

$(y=mx)$ $m=1$

$\tan\theta=1$

$\boxed{\theta=\pi/4}$

$\theta: \pi/4 \rightarrow \pi/2$.

$$\therefore \int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} \frac{1}{r}\cos\theta.\,r\,dr\,d\theta.$$

$$= \sqrt{2}-1$$

⑦. By changing into polar coordinates, Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$.

Sol:- change to polar co-ordinates by putting

$x=r\cos\theta$

$y=r\sin\theta$

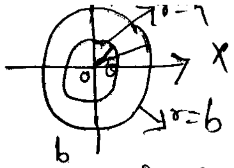
$dx\,dy=r\,dr\,d\theta$

Now $x^2+y^2=a^2 \Rightarrow r^2=a^2 \Rightarrow \boxed{r=a}$

$x^2+y^2=b^2 \Rightarrow r^2=b^2 \Rightarrow \boxed{r=b}$

$\therefore r: a \rightarrow b$.

$\therefore$ and $\theta : 0 \to 2\pi$



Hence $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^2(\cos^2\theta+\sin^2\theta)}\, r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3 \cos^2\theta\sin^2\theta \,dr\,d\theta$$

$$= \int_{0}^{2\pi} \left(\frac{r^4}{4}\right)_a^b \frac{\sin^2 2\theta}{4}\,d\theta$$

$$= \left(\frac{b^4}{4}-\frac{a^4}{4}\right)\frac{1}{4}\int_0^{2\pi} \sin^2 2\theta\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta$$

$$= \frac{b^4-a^4}{16}\left(\theta - \frac{\sin 4\theta}{4}\right)_0^{2\pi}$$

$$= \frac{b^4-a^4}{32}(2\pi) = \frac{\pi(b^4-a^4)}{16}$$

(8). Transform the following to cartesian form and hence Evaluate $\int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\,dr\,d\theta$.

Sol:- The region of Integration is given by $r=0$, $r=a$ & $\theta=0$, $\theta=\pi$.

In the cartesian coordinates the same region is given by $x=0$, $y=0$ ($\because r=0$) and $x^2+y^2=a^2$ ($\because r=a$)

Since $\theta: 0\to\pi$, the region of Integration is the semi circle $x^2+y^2=a^2$

Let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

$$\therefore \int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\, d\theta = \int_{\theta=0}\int_{r=0}^{a} (r\sin\theta)(r\cos\theta)(r\, dr\, d\theta)$$

$$= \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy \, dx\, dy$$

$$= \int_{-a}^{a} x \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} \frac{x}{2}(a^2-x^2)\, dx$$

$$= 0.$$

9. By using the transformation $x+y=u$, $y=uv$ Hence S.T

$$\int_0^1\int_0^{1-x} e^{y/x+y}\, dy\, dx = \frac{1}{2}(e-1).$$

Sol: The region of integration is given by.

$y=0$, $y=1-x$; $x=0$ & $x=1$.

Given transformation is

$x+y=u$ —— (1)

$y=uv$ —— (2)

(2) in (1) $\Rightarrow x+uv=u \Rightarrow x=u(1-v)$ —— (3)

Now $\Rightarrow y=0 \Rightarrow uv=0 \Rightarrow \boxed{u=0 \text{ (or) } v=0}$ $[\because (2)]$

$y=1-x \Rightarrow \cancel{x}+1-\cancel{x}=u \Rightarrow \boxed{u=1}$ $[\because (1)]$

$x=0 \Rightarrow \boxed{u=0 \text{ (or) } v=1}$ $[\because (3)]$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u.$$

$$\therefore \int_0^1\int_0^{1-x} e^{y/x+y}\, dy\, dx = \iint_R e^{\frac{uv}{u}} |J|\, du\, dv.$$

$$= \iint e^{v} |J|\, du\, dv.$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} e^{v}\, u\, du\, dv = \frac{1}{2}(e-1)$$

(10). Evaluate $\iint_R (x+y)^2\,dx\,dy$ where 'R' is the Parallelogram in xy-plane with vertices $(1,0)$, $(3,1)$, $(2,2)$ & $(0,1)$ by using the transformations $u=x+y$ and $v=x-2y$.

Sol:- $u=x+y$ —(1) $\qquad v=x-2y$ —(2)

(1) − (2) $\Rightarrow$ $u-v=3y \Rightarrow y=\frac{1}{3}(u-v)$. —(3)

(1) $\Rightarrow$ $x=u-y$

$=u-\frac{1}{3}(u-v)$

$=\dfrac{3u-u+v}{3}$

$x=\frac{1}{3}(2u+v)$ —(4)

Now $A(1,0)$ i.e, $x=1, y=0 \Rightarrow \boxed{u=1,\ v=1}$

$B(0,1)$ i.e, $x=0, y=1 \Rightarrow \boxed{u=1,\ v=-2}$

$C(2,2)$ i.e, $x=2, y=2 \Rightarrow \boxed{u=4,\ v=-2}$

$D(0,1)$ i.e, $x=0, y=1 \Rightarrow \boxed{u=4,\ v=1}$

$\therefore \iint (x+y)^2\,dx\,dy = \iint_{u\,v} u^2\,|J|\,du\,dv$

$= \int_1^4 \int_{-2}^{1} u^2\,\frac{1}{3}\,du\,dv$

$= 21$ //

$|J| = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix}$

$= -1/3$ //

(11). By changing into polar-co-ordinates Evaluate

$$\int_0^\infty \int_0^\infty \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy.$$

Sol:- Put $x=r\cos\theta$, $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

$\theta: 0 \to \pi/2$

$r: 0 \to \infty$.

$$G.I = \int_0^{\pi/2} \int_{r=0}^{\infty} \frac{r^2 \cos^2\theta}{r^3} \, r \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\infty} \left(\theta + \frac{\sin 2\theta}{2} \right)_0^{\pi/2} dr$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} \int_0^{\infty} dr.$$

$$= \frac{\pi}{4} \cdot \lim_{P \to \infty} \int_0^{P} dr$$

$$= \frac{\pi}{4} \lim_{P \to \infty} (r)_0^{P} = \frac{\pi}{4} \lim_{P \to \infty} (P) = \infty$$

Problems

by change of order of Integration :-

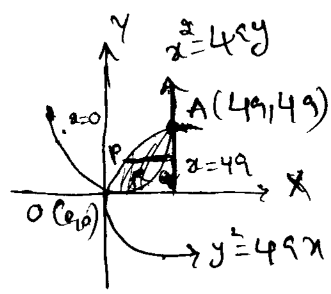
(1). Evaluate $\int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy \, dx$

Sol:- the limits are $\Rightarrow x=0, x=4a$

$\Rightarrow y = \frac{x^2}{4a} \; ; y = 2\sqrt{ax}$

$\Rightarrow x^2 = 4ay \; ; \; y^2 = 4ax.$

$\Rightarrow y = \frac{x^2}{4a} \; ; \; y = 2\sqrt{ax}.$



we get $\frac{x^2}{4a} = 2\sqrt{ax} \Rightarrow x^2 = 8a\sqrt{ax}$

$\Rightarrow x^4 = 64a^2 ax \Rightarrow x^4 - 64a^3 x = 0 \Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0 \;\; \Big| \;\; x^3 = 64a^3$
$x^3 = (4a)^3$
$x = 4a.$

$\therefore x : 0 \to 4a.$

put $x=0 \Rightarrow y=0$

$x = 4a \Rightarrow y = 4a$

$\therefore y : 0 \to 4a.$

$\therefore$ The Point of integration is $O = (0,0)$
$A = (4a, 4a).$

Here 'R' is the region of Integration and the limits of 'y' are function of 'x'. Draw strips PQ parallel to x-axis and Fix 'y'. For fixed 'y' the limits are

$x: \frac{y^2}{4a} \to 2\sqrt{ay}.$

$y: 0 \to 4a$

$$\therefore R.I = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dx\, dy = \int_0^{4a} (x)_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy = \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left(2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{1}{4a} \frac{y^3}{3}\right)_{y=0}^{4a} = \left(\frac{4}{3}\sqrt{a}\, y^{3/2} - \frac{1}{12a} y^3\right)_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{1}{12a}(4a)^3 = \frac{4}{3} a^2 \times 8 - \frac{1}{\cancel{12a}_3} \overset{16}{\cancel{64}} a^{\cancel{3}2}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{a^2}{3}[16] \,//$$

Change of order of Integration :- In the given double Integral if the limits are constant. Then there is no need to change order of Integration.

But the limits of integration are variable then the change of order of Integration requires the changes in limits.

In this process we have to draw a sketch of region and we have to change limits according to region.

Case (i) :- working rule to change order of Integration in given double integral.

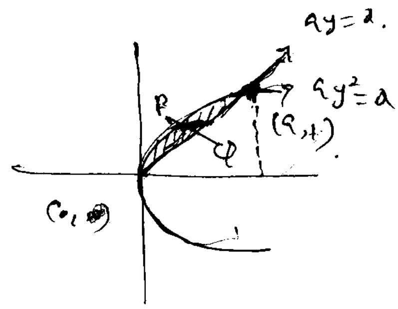
$$G.I = \int_{x=a}^{b} \int_{y=f_1(x)}^{y_2=f_2(x)} f(x,y)\, dy\, dx.$$

draw the region by taking the given curve Eqn $y=f_1(x)$; $y=f_2(x)$ & the lines $x=a$, $x=b$.

First draw the strips parallel to x-axis and fix 'y'. (24)
For fixed 'y' we have to take the limits of 'x' in terms of 'y'.
the limits for 'y' as constant.
If there is any change in the direction of strips, then divide the region into sub-region and in each subregion to find the limits of 'x' & 'y'.

$\therefore$ The R.I is $\displaystyle\int_{y=c}^{d}\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy$.

(2). Evaluate $\displaystyle\int_{x=0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy$



Sol:- the limits are $x=0,\ x=a$

$y=\frac{x}{a}$ & $y=\sqrt{\frac{x}{a}}$

$\Rightarrow x=ay$ & $ay^2=x$

we get $\frac{x}{a}=\sqrt{\frac{x}{a}} \Rightarrow \frac{x^2}{a^2}=\frac{x}{a} \Rightarrow x=0 \rightarrow a$

put $x=0 \rightarrow y=0$

$x=a \Rightarrow y=1$.

$\therefore$ the points of Integration is $O=(0,0)$
$A=(a,1)$

Here 'R' is the region of Integration and the limits of 'y' are function of 'x'. Draw the strips parallel to x-axis and fix 'y'. For fixed 'y' the limits are

$y: 0 \rightarrow 1$

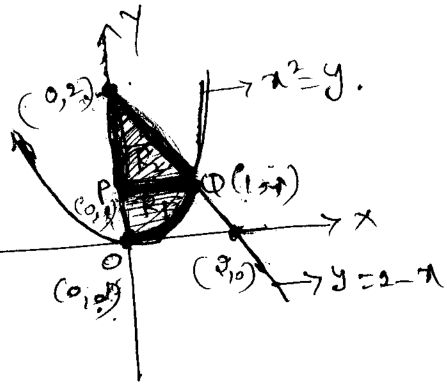
$x: ay^2 \rightarrow ay$.

$\therefore \displaystyle\int_{x=0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy = \int_{0}^{1}\int_{x=ay^2}^{ay} (x^2+y^2)\,dx\,dy = \int_{y=0}^{1}\left(\frac{x^3}{3}+y^2x\right)_{x=ay^2}^{ay} dy$

$= \displaystyle\int\left(\frac{a^3y^3}{3}+ay^3-\frac{a^3y^6}{3}-ay^4\right)dy = \left(\frac{a^3}{3}\frac{y^4}{4}+\frac{ay^4}{4}-\frac{a^3}{3}\frac{y^7}{7}-\frac{ay^5}{5}\right)_0^1 = \frac{a^3}{12}+\frac{a}{4}-\frac{a^3}{21}-\frac{a}{5}$

$= a^3/28 + a/20$

3. change of order of integration $\int_0 \int_{x^2} xy\,dx\,dy$ and hence Evaluate the double integral.



Sol:- the limits are $x=0$, $x=1$.

$y=x^2$, $y=2-x$.

$\Rightarrow x+y=2$

$\Rightarrow \frac{x}{2}+\frac{y}{2}=1$

we get

$\Rightarrow x^2=2-x$

$\Rightarrow x^2+x-2=0$

$\Rightarrow x^2+2x-x-2=0$

$\Rightarrow x(x+2)-(x+2)=0$

$x=-2, 1.$

Put $x=1 \Rightarrow y=2-1$
$=1$

$\therefore \boxed{x=1 \ \& \ y=1}$

R_1:- $x: 0 \to \sqrt{y}$
$y: 0 \to 1$

R_2:- $x: 0 \to 2-y.$
$y: 1 \to 2$

$$\therefore R.I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{x=0}^{2-y}\int_{y=1}^{2} xy\,dx\,dy$$

$$= \int_{y=0}^{1}\left(\frac{x^2}{2}y\right)_0^{\sqrt{y}} dy + \int_{y=1}^{2}\left(\frac{x^2}{2}y\right)_0^{2-y} dy.$$

$$= \int_{y=0}^{1}\frac{y^2}{2}\,dy + \int_{y=1}^{2}\frac{(2-y)^2}{2}\,y\,dy$$

$$= \int_{y=0}^{1}\frac{y^2}{2}\,dy + \int_{y=1}^{2}\left(\frac{4+y^2-4y}{2}\right)y\,dy$$

$$= \left(\frac{y^3}{6}\right)_0^1 + \frac{1}{2}\left(4\frac{y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}\right)_1^2$$

$$= \frac{1}{6}+\frac{1}{2}\left[2(4)+4-\frac{4}{3}(8)-\left(\frac{4}{2}+\frac{1}{4}-\frac{4}{3}\right)\right] \quad (0.2)\ \frac{1}{6}+\frac{1}{2}(8+4-10.66-2+0.25+1.33).$$

$= 9/24$

$= 3/8 = 0.375$

$= \frac{1}{6}+\frac{1}{2}(0.22)$

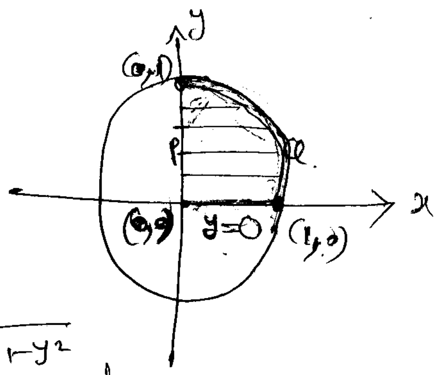
$= \frac{1}{6}$ [redacted] $= 0.37$

(4). $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \pi/16$

Sol:- the limits are $x=0$, $x=1$

$y=0$, $y=\sqrt{1-x^2}$

$x^2+y^2=1$



$$R.I = \int_{y=0}^{1}\left[\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\right]dy = \int_0^1 \left(y^2 x\right)_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \left(y^2\sqrt{1-y^2}\right)dy$$

put $y=\sin\theta$

$dy = \cos\theta\,d\theta$.

limits: at $y=0 \Rightarrow \theta=0$

$y=1 \Rightarrow \theta=\pi/2$

$$\therefore R.I = \int_0^{\pi/2} \sin^2\theta\sqrt{\cos^2\theta}\,d\theta\cos\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{2\sin\theta\cos\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \sin^2 2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)_0^{\pi/2}$$

$$= \frac{1}{8}\cdot\frac{\pi}{2} = \pi/16 \,//$$

Case(2):- change of order of Integration $\int_{y=a}^{b}\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy$

Draw the region of Integration by using given limits

$y=a$; $y=b$

$x=f_1(y)$; $x=f_2(y)$

(27)

∵ Draw the strips $\parallel^{el}$ to Y-axis and keep 'x' as fix.

For fixed 'x' take limits of 'y' in terms of 'x' & the limits of 'x' as constant.

∴ By changing order of integration

$$R.I = \int_{x=a}^{b} \int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\, dx\, dy.$$

(1). $$\int_{y=-a}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy.$$

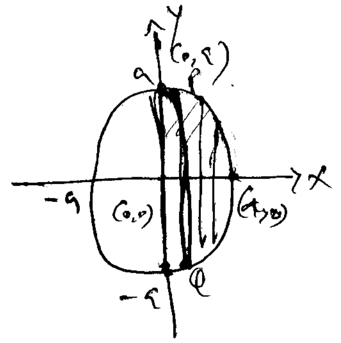
Sol: limits are $y=-a$, $y=a$

$x=0$, $x=\sqrt{a^2-y^2}$

$x^2=a^2-y^2$

$\Rightarrow x^2+y^2=a^2$

$\boxed{r=a}$



∴ $x: 0 \to a$

$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

$$R.I = \int_{x=0}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\, dy\, dx$$

(2). Evaluate $$\int_{y=0}^{b} \int_{x=0}^{\frac{a\sqrt{b^2-y^2}}{b}} xy\, dx\, dy.$$

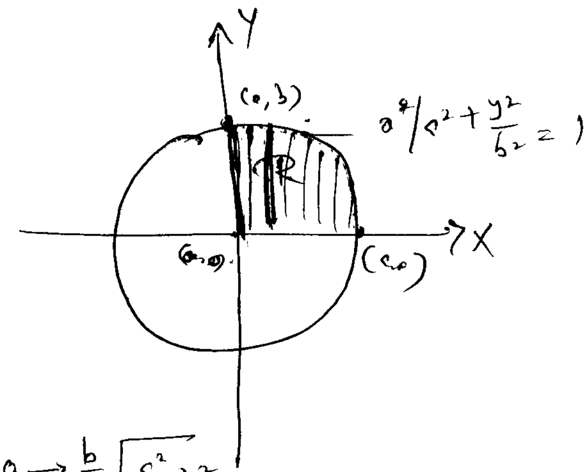
Sol:- $y=0$, $y=b$.

$x=0$, $x=\dfrac{a\sqrt{b^2-y^2}}{b}$.

$b^2x^2=a^2b^2-a^2y^2$

$\Rightarrow b^2x^2+y^2a^2=a^2b^2$

$\Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$



∴ $y: 0 \to \frac{b}{a}\sqrt{a^2-x^2}$

$x: 0 \to a$.

(Ans: $\frac{a^2b^2}{8}$)

3. $\int_0^1 \int_x^{\sqrt{2-x^2}} f(x,y)\, dy\, dx$.

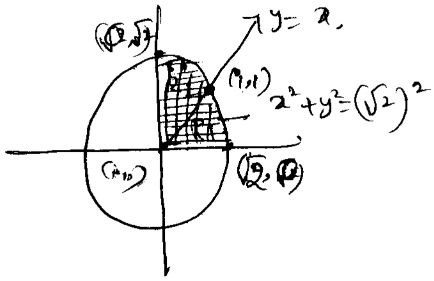
Sol:- $x=0$, $x=1$

$y=x$, $y=\sqrt{2-x^2}$

$y^2=2-x^2$

$x^2+y^2=(\sqrt{2})^2$.

$\boxed{r=\sqrt{2}}$



$\Rightarrow x=\sqrt{2-x^2}$

$x^2=2-x^2$

$2x^2=2$

$\boxed{x=\pm 1}$

put $x=1 \Rightarrow y=1 \Rightarrow (1,1)$

$x=-1 \Rightarrow y=-1 \Rightarrow (-1,-1)$

$\therefore R_1:$ $y: 0 \to 1$

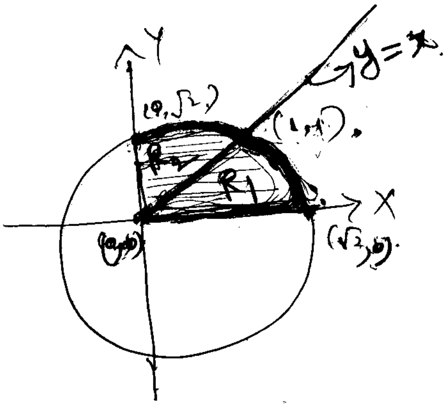
$x: 0 \to y$

$R_2:$ $y: 1 \to \sqrt{2}$

$x: 0 \to \sqrt{2-y^2}$

$\therefore R.I = \int_{y=0}^{1} \int_{x=0}^{y} f(x,y)\, dx\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} f(x,y)\, dx\, dy$

4. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$



$R_1:$ $x: 0 \to y$

$y: 0 \to 1$

$R_2:$ $x: 0 \to \sqrt{2-y^2}$

$y: 1 \to \sqrt{2}$

Sol:- $x=0$, $x=1$

$y=x$, $y=\sqrt{2-x^2}$

$x^2+y^2=2$

Ans: $1-\frac{1}{\sqrt{2}}$

5. $\int_0^a \int_{\sqrt{ax}}^{a} \frac{y^2\, dy\, dx}{\sqrt{y^4-a^2x^2}}$

Sol:- $x=0$; $y=\sqrt{ax} \Rightarrow y^2=ax$

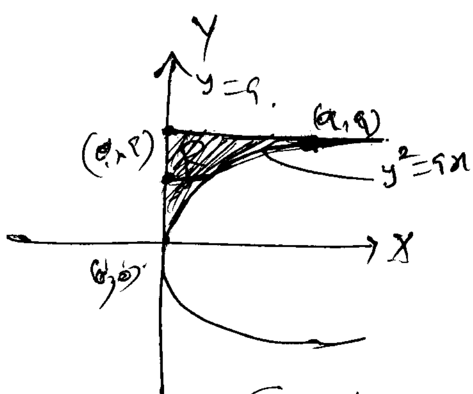
$x=a$; $y=a$. $\Rightarrow x=\frac{y^2}{a}$.

$ax=a^2$

$\to a^2-ax=0$

$\therefore$ $y: 0 \to a$.

$x: 0 \to y^2/a$.



($\therefore \pi a^2/6$)

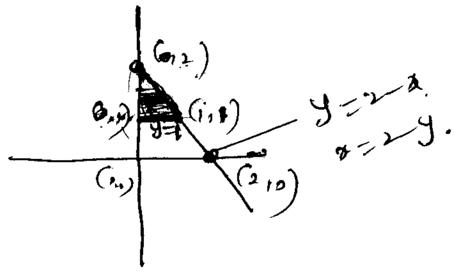
6. $\int_0^1 \int_1^{2-x} xy \, dy \, dx$

Sol:- $x=0,\ x=1$

$y=1,\ y=2-x$

$\rightarrow x+y=2$

$\Rightarrow \frac{x}{2}+\frac{y}{2}=1$



Let $2-x=1$

$x=1$.

$\therefore (x,y)=(1,1)$.

$\therefore\ y: 1 \rightarrow 2$.

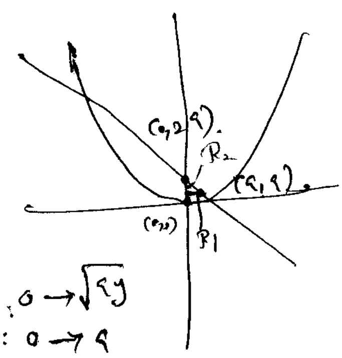
$x: 0 \rightarrow 2-y$.

$5/24$.

7. $\int_0^a \int_{x^2/a}^{2a-x} xy^2 \, dy \, dx$.

Sol:- $x=0$; $y=\frac{x^2}{a} \Rightarrow x^2=ay$

$x=a$; $y=2a-x \Rightarrow$



$\frac{x^2}{a}=2a-x$.

$x^2=2a^2-xa$

$\Rightarrow x^2+ax-2a^2=0$

$\Rightarrow x=\frac{-a\pm\sqrt{a^2+8a^2}}{2}$

$x=\frac{-a\pm 3a}{2} \Rightarrow x=a$,

$(x=-2a)$. $\times$

Put $x=a \Rightarrow y=a \Rightarrow (a,a)$

$x=-2a \Rightarrow y=$ doesn't take.

$\therefore R_1:$ $x: 0 \rightarrow \sqrt{ay}$
$y: 0 \rightarrow a$

$R_2:$ $x: 0 \rightarrow 2a-y$.
$y: a \rightarrow 2a$

Ans:- $\frac{47a^5}{120}$ //.

TRIPLE INTEGRAL

Let $f(x,y,z)$ is a continuous function defined on finite region 'v', then the triple integral is denoted by

$$\iiint_V f(x,y,z)\,dv.$$

where $dv = dx\,dy\,dz$.

①. Evaluate $\int_{z=0}^{1}\int_{y=1}^{2}\int_{x=1}^{3} xyz\,dx\,dy\,dz$.

Sol:- $\int_{z=0}^{1}\int_{y=1}^{2} yz\left(\frac{x^2}{2}\right)_1^3 dy\,dz = \frac{1}{2}\int_{z=0}^{1}\int_{y=1}^{2} yz\,(9-4)\,dy\,dz$

$= \frac{1}{2}\int_{z=0}^{1}\int_{y=1}^{2} yz\,(5)\,dy\,dz = \frac{5}{2}\int_{z=0}^{1}\left(y^2/_2\right)_1^2 z\,dz = \frac{5}{4}\int_0^1 (4-1)z\,dz$

$= \frac{15}{4}\left(\left(\frac{z^2}{2}\right)\right)_0^1 \Rightarrow \frac{15}{8}(1-0) = 15/8$ //

②. Evaluate $\int_{y=0}^{1}\int_{x=y}^{1}\int_{z=0}^{1-x} x\,dz\,dx\,dy$. $(1/_{12})$

③. Evaluate $\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dx\,dy\,dz$. $(1/_6)$

④. $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$ $(1/_{48})$

⑤. $\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\,dx\,dy\,dz$. (0)

⑥. $\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta$ $\left(\frac{5\pi a^4}{128}\right)$

Areas

①. Find the area enclosed by parabola $x^2=y$; $y^2=x$.

Sol:- $y=x^2$; $x^2=y$.

$\therefore \Rightarrow (x^2)^2 = x \Rightarrow x^4 - x = 0$.
$\Rightarrow x = 0, 1$

put $x=0 \Rightarrow y=0$ $\qquad (1/3)$.
$x = 1 \Rightarrow y = 1$.

$\therefore x : 0 \to 1$.
$y : x^2 \to \sqrt{x}$.

(2). $y = 4x - x^2$ and $y = x$.

sol:- $4x - x^2 = x \Rightarrow x = 0, 3$.

put $x=0 \Rightarrow y=0$
$x = 3 \Rightarrow y = 3$. $\qquad (9/2)$.

$\therefore x : 0 \to 3$.
$y : x \to 4x - x^2$.

(3). $y^2 = 4ax$, $x^2 = 4ay$.

sol:- $y^2 = 4a(2\sqrt{ay})$
$\Rightarrow y^2 = 8a^{3/2} y^{1/2}$
$\Rightarrow y^2 - 8a^{3/2} y^{1/2} = 0$
$\Rightarrow y^{1/2}\left[y^{3/2} - 8a^{3/2}\right]$

(or) $y^2 = 4ax$; $x^2 = 4ay$.
$x = \frac{y^2}{4a}$; $\frac{y^4}{16a^2} = 4ay$
$y^3 = 64a^3$.
$\boxed{y = 4a}$

put $y = 0 \Rightarrow x = 0$
$y = 4a \Rightarrow x = 4a$.

$\therefore x : 0 \to 4a$
$y : \frac{x^2}{4a} \to 2\sqrt{ax}$.

(4). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\left(\frac{\pi ab}{4}\right)$.

sol:- $x : 0 \to a$
$y : 0 \to \frac{b}{a}\sqrt{a^2 - x^2}$

(5). $y = x^2$; $y = x$.

$x : 0 \to 1$ $\qquad (1/6)$
$y : x^2 \to x$

Finding the Volume apply to Triple Integral :-

the volume of a solid is given by $\iiint_V dx\,dy\,dz$,

(1). find the volume of the tetrahedron bounded by the planes

$x=0;\ y=0;\ z=0;\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Ans: Here. $\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}$

$z=c\left[1-\frac{x}{a}-\frac{y}{b}\right]$

$z:0 \to c\left[1-\frac{x}{a}-\frac{y}{b}\right]$

put $z=0$

$\frac{x}{a}+\frac{y}{b}=1$

$\frac{y}{b}=1-\frac{x}{a}$

$y=b\left(1-\frac{x}{a}\right)$

$\therefore y:0 \to b\left[1-\frac{x}{a}\right]$

$x:0 \to a.$

$$V=\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c[1-x/a-y/b]} dz\,dy\,dx. = \frac{abc}{6}$$

(2). Evaluate $\iiint_V dx\,dy\,dz$, where 'v' is the finite region of space formed by the planes $x=0, y=0, z=0$ & $2x+3y+4z=12$. (: 12)

(3). Find the volume common to the cylinders $x^2+y^2=a^2$ & $x^2+z^2=a^2$
→(1) →(2)

Ans:- (2) $\Rightarrow z^2=a^2-x^2.$

$z: -\sqrt{a^2-x^2} \to +\sqrt{a^2-x^2}$

$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}.$

$x: -a \to +a.$

$\left(\frac{16a^3}{3}\right)$

Change of variables from coordinates to spherical polar coordinates (33)

Sub: $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \frac{\partial x}{\partial\phi} \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix}$$

Note: 1). Case (1): For whole volume of sphere take limits

$r: 0 \to a$

$\theta: 0 \to \pi$

$\phi: 0 \to 2\pi$.

2). In the 1st octant $Oxyz$ take limits of volume of a sphere

$r: 0 \to a$

$\theta: 0 \to \pi/2$

$\phi: 0 \to \pi/2$

* Problems *

①. $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ take a over the volume enclosed by the sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates.

Sol: G.I $= \iiint (x^2+y^2+z^2)\,dx\,dy\,dz$

Sub: $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

For whole volume of the sphere $x^2+y^2+z^2=1^2$.

Limits are: $r: 0 \to 1$ $\quad (4\pi/5)$

$\theta: \to \pi$

$\phi: \dots$

Q. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical Polar coordinates.

Sol: limits are $z=0$ & $z=\sqrt{1-x^2-y^2}$

$x^2+y^2+z^2=1$

$r: 0 \to 1$

$\theta: 0 \to \pi/2$

$\phi: 0 \to \pi/2$.

$$\text{G.I} = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{r^2 \sin\theta \, dr\, d\theta\, d\phi}{\sqrt{1-r^2}}$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \left\{ \int_{r=0}^{1} \frac{1-(1-r^2)}{\sqrt{1-r^2}}\, dr \right\} \sin\theta \, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \left\{ \sin^{-1} r - \frac{r}{2}\sqrt{1-r^2} - \frac{1}{2}\sin^{-1} r \right\}_0^1 \sin\theta \; d\phi\, d\theta.$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2} \right) \sin\theta \; d\theta\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \, d\theta\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \left(-\cos\theta \right)_0^{\pi/2} d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} (-0+1)\, d\phi$$

$$= \frac{\pi}{4} \cdot \pi/2$$

$$= \pi^2/8$$

. change of variable from cartesian coordinates to cylindrical coordinates.

Sol:- G.I = $\iiint_V f(x,y,z)\,dx\,dy\,dz$

Sub, $x = r\cos\theta$

$y = r\sin\theta$

$z = z$.

$dx\,dy\,dz = |J|\,dr\,d\theta\,dz$.

$\therefore |J| = \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta$

$= r$.

$\therefore \iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz$.

(4). By transforming into cylindrical coordinates. Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$. taken order of the region of $0 \le z \le x^2+y^2 \le 1$.

Sol:- G.I $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ to change into cylindrical coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$.

$dx\,dy\,dz = r\,dr\,d\theta\,dz$.

Here the region is bounded by.

$0 \le z \le 1$ & $0 \le x^2+y^2 \le 1$.

$z: 0 \to 1$ & $x^2+y^2 = 1$.

$r: 0 \to 1$

$\theta: 0 \to 2\pi$.

$\left(A: \frac{5\pi}{6}\right)$

$\boxed{5}$. Centre of Gravity :-

Let $(\bar{x}, \bar{y})$ be the centroid of the region. Since the region is symmetric about x-axis i.e. $\bar{y} = 0$.

$\therefore \bar{x} = \left.\frac{\iint x\,dx\,dy}{\iint dx\,dy}\right]_4$ no need.

UNIT - III

Extra Problems

① Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol $\int_{y=0}^{1}\left[\int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2}}\right] \frac{dy}{\sqrt{1-y^2}} = \int_{y=0}^{1} \left(\sin^{-1}x\right)_{x=0}^{1} \frac{dy}{\sqrt{1-y^2}}$

$= \int_{y=0}^{1} \pi/2 \cdot \frac{dy}{\sqrt{1-y^2}}$

$= \frac{\pi}{2}\left(\sin^{-1}y\right)_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$,

② $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dy\,dx}{1+x^2+y^2}$

Sol $\int_{x=0}^{1} \int_{y=0}^{\sqrt{(1+x^2)}} \frac{dy\,dx}{(1+x^2)+y^2}$

$\int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{(1+x^2)}} \frac{dy}{(1+x^2)+y^2}\right] dx = \int_0^1 \left[\int_{y=0}^{\sqrt{(1+x^2)}} \frac{1}{\left(\sqrt{(1+x^2)}\right)^2+y^2} dy\right] dx$

$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{y=\sqrt{1+x^2}} dx$

$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - \tan^{-1}0 . \frac{1}{\sqrt{1+x^2}}\right) dx$

$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \pi/4 \, dx$

$= \frac{\pi}{4} \left[\log\left(x+\sqrt{x^2+1}\right)\right]_{x=0}^{1}$ (or) $= \frac{\pi}{4}\left(\sinh^{-1}x\right)_0^1$

$= \frac{\pi}{4}\log(1+\sqrt{2})$

$= \frac{\pi}{4}\sinh^{-1} 1$

3). $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.

Sol $\Rightarrow \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^\infty\int_0^\infty e^{-y^2}\left[e^{-x^2}dx\right]dy$

$$= \int_0^\infty e^{-y^2}\left(\int_0^\infty e^{-x^2}dx\right)dy \qquad \left(\because \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\right)$$

$$= \int_0^\infty e^{-y^2}\left(\sqrt{\pi}/2\right)dy$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2} = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4} \;/\!/$$

④. $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$

Sol: $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx = \int_{x=0}^a\left[\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2-y^2}\,dy\right]dx$

$\left(\sqrt{a^2-x^2} \text{ formulae}\right)$

$$= \int_{x=0}^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^a \left[\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-(a^2-x^2)} + \frac{a^2-x^2}{2}\sin^{-1}(1)\right]dx$$

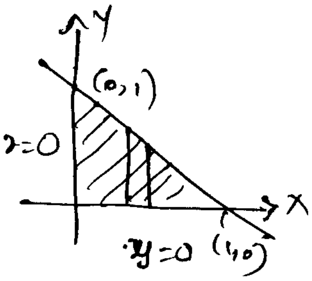
$$= \int_{x=0}^a \left[\frac{\sqrt{a^2-x^2}}{2}(0) + \frac{a^2-x^2}{2}\sin^{-1}(1)\right]dx$$

$$= \int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}(1)\right]dx$$

$$= \frac{\pi}{4}\int_0^a (a^2-x^2)\,dx$$

$$= \frac{\pi}{4}\left(a^2x-\frac{x^3}{3}\right)_0^a = \frac{\pi}{4}\left(a^3-\frac{a^3}{3}\right) = \frac{\pi a^3}{6} \;/\!/$$

5. Evaluate $\iint (x^2+y^2)\,dx\,dy$ in the +ve quadrant for which $x+y \le 1$.



Sol:
$$\int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\,dx\,dy = 1/6.$$

6. Evaluate $\iint x^2\,dx\,dy$ over the region bounded by Hyperbola $xy=4$, $y=0$, $x=1$ & $x=4$.

Sol: the limits for y are $y=0$ to $y=4/x$

" " x " $x=1$ to $x=4$.

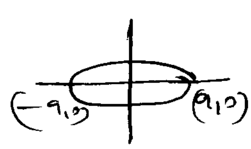
$$\therefore \int_{x=1}^{4}\int_{y=0}^{4/x} x^2\,dx\,dy = 30.$$

7. $\iint_R y\,dx\,dy$ bounded by Y-axis the curve $y=x^2$ and the line $x+y=2$ in the 1st quadrant.

Sol:
$$\left.\begin{aligned} x&=0\\ y&=x^2\\ x+y&=2 \end{aligned}\right\} \Rightarrow \begin{aligned} x^2&=2-x\\ x^2+x-2&=0 \end{aligned} \Rightarrow x^2+2x-x-2=0 \Rightarrow x(x+2)-(x+2)=0$$
$$x=-2,1.$$

$$\therefore \int_{x=0}^{1}\int_{y=x^2}^{2-x} y\,dx\,dy = 16/15.$$

8. $\iint (x^2+y^2)\,dx\,dy$ over the Area bounded by the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Sol:
$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

$x=\pm a.$

$$\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy = \frac{\pi ab}{4}(a^2+b^2)$$
(put $x = a\sin\theta$)

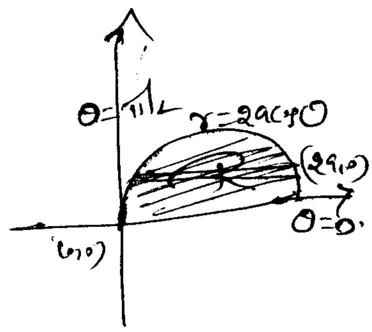
9). $\iint xy\,dx\,dy$ taken over the +ve quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (4)

Sol:
$$\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dx\,dy = \frac{a^2b^2}{8}$$

10). S.T $\iint r^2\sin\theta\,dr\,d\theta = \frac{2a^3}{3}$, where 'R' is the Semi-circle $r=2a\cos\theta$ above the initial line.

Sol:
$r=0$
$r=2a\cos\theta$

& Put $r=0 \Rightarrow 0=2a\cos\theta$
$\cos\theta=0$
$\theta=\pi/2$.



$r=0 \rightarrow 2a\cos\theta$

$\therefore \theta: 0 \rightarrow \pi/2$

$$\therefore \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2a\cos\theta}\sin\theta\,d\theta$$

$$=\frac{1}{3}\int_{\theta=0}^{\pi/2} 8a^3\cos^3\theta\,\sin\theta\,d\theta = -\frac{8a^3}{3}\int_{\theta=0}^{\pi/2}\cos^3\theta(-\sin\theta)\,d\theta$$

$$= -\frac{8a^3}{3}\left[\frac{(\cos\theta)^{3+1}}{3+1}\right]_{\theta=0}^{\pi/2}$$

$$\left[\because \int [f'(x)][f(x)]^n dx = \frac{f(x)^{n+1}}{n+1}+C\right]$$

$$= \frac{-8a^3}{12}\left(\cos^4\theta\right)_0^{\pi/2}$$

$$= \frac{-2a^3}{3}\left[\cos^4(\pi/2)-\cos^4(0)\right]$$

$$= \frac{-2a^3}{3}[0-1]$$

$$= \frac{2a^3}{3} \;//$$

(11). $\iint r^3 \, dr \, d\theta$ over the area included between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$.

Sol: $r : 2\sin\theta \to 4\sin\theta$ $\qquad$ $4\sin\theta = 2\sin\theta$

$\theta : 0 \to \pi$ $\qquad$ $2\sin\theta - \sin\theta = 0$

$\sin\theta = 0$

$\boxed{\theta = 0, \pi}$

$$\therefore \int_{\theta=0}^{\pi} \left[\int_{r=2\sin\theta}^{4\sin\theta} r^3 \, dr \right] d\theta = \int_{\theta=0}^{\pi} \left(\frac{r^4}{4} \right)_{r=2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi} \left(256\sin^4\theta - 16\sin^4\theta \right) d\theta$$

$$= \frac{240}{4} \int_{\theta=0}^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \int_{\theta=0}^{\pi} \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta \qquad (\because \cos 2\theta = 1 - 2\sin^2\theta)$$

$$= \frac{60}{4} \int_{\theta=0}^{\pi} \left(1 + \cos^2 2\theta - 2\cos 2\theta \right) d\theta$$

$(\because \cos 2\theta = 2\cos^2\theta - 1)$

$$= \frac{30}{2} \int_{\theta=0}^{\pi} \left[1 + \left(\frac{1+\cos 4\theta}{2} \right) - 2\cos 2\theta \right] d\theta$$

$$= \frac{15}{2} \int_{\theta=0}^{\pi} \left(3 + \cos 4\theta - 4\cos 2\theta \right) d\theta$$

$$= \frac{15}{2} \left(3\theta + \frac{\sin 4\theta}{4} - \frac{4\sin 2\theta}{2} \right)_{\theta=0}^{\pi}$$

$$= \frac{15}{2} (3\pi)$$

$$= \frac{45\pi}{2} \,//$$

(12). $\iint r\sin\theta \, dr\, d\theta$ over the cardioid $r = a(1+\cos\theta)$ above the Initial line.

Sol:- $r: 0 \to a(1+\cos\theta)$

$\theta = 0 \to \pi$

Put $r=0 \Rightarrow 0 = a(1+\cos\theta)$

$\Rightarrow \cos\theta = -1$

$\theta = \pi$

$$I = \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a(1+\cos\theta)} r\, dr\right] d\theta \sin\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta = -\frac{a^2}{2}\int_{\theta=0}^{\pi}(1+\cos\theta)^2(-\sin\theta)\, d\theta$$

$$= -\frac{a^2}{2}\left[\frac{(1+\cos\theta)^3}{3}\right]_{\theta=0}^{\pi}$$

$$\left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= -\frac{a^2}{6}\left[(1+\cos\pi)^3 - (1+\cos 0)^3\right]$$

$$= -\frac{a^2}{6}\left[(1-1)^3 - (2)^3\right]$$

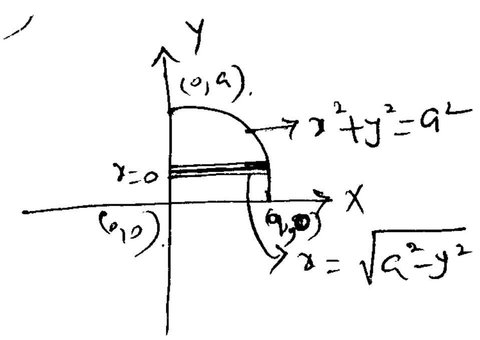
$$= \frac{2^3 a^2}{6}$$

$$= \frac{4a^2}{3}$$

(13). By changing the order of integration Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

Ans:- $$\int_{x=0}^{a} \int_{y=0}^{y=\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx.$$



For fixed 'y', the limits are.

$y : 0 \to a$

$x : 0 \to \sqrt{a^2-y^2}$ (43)

$$\therefore R.I = \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{\underbrace{\left(\sqrt{(a^2-y^2)}\right)^2 - x^2}_{(a^2-x^2)}}\, dx \right] dy$$

$\boxed{\because} \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$

$$= \int_{y=0}^{a} \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{(a^2-y^2)}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}} \right]_{x=0}^{\sqrt{a^2-y^2}} dy$$

$$= \int_{y=0}^{a} \left(\frac{\sqrt{a^2-y^2}}{2}\sqrt{a^2-y^2-a^2+y^2} + \frac{(a^2-y^2)}{2}\frac{\pi}{2} - 0 \right) dy$$

$$= \int_{y=0}^{a} \left(\frac{\sqrt{a^2-y^2}}{2}(0) + \left(\frac{a^2-y^2}{2}\right)\frac{\pi}{2} \right) dy$$

$$= \frac{\pi}{4}\int_{y=0}^{a} (a^2-y^2)\, dy = \frac{\pi}{4}\left(a^2y - \frac{y^3}{3}\right)_{y=0}^{a}$$

$$= \frac{\pi}{4}\left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{\pi}{\cancel{4}_2}\left(\frac{\cancel{2}a^3}{3}\right)$$

$$= \frac{\pi a^3}{6}$$

//

(14). change of order of Integration and Evaluate $\int_0^b \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy \, dx \, dy$. (44)

Sol: G.I $= \int_{y=0}^{b} \int_{x=0}^{\frac{a\sqrt{b^2-y^2}}{b}} xy \, dx \, dy$.

Given limits are $x=0$ and $x = \frac{a\sqrt{b^2-y^2}}{b}$

$\Rightarrow x^2b^2 = a^2(b^2-y^2)$

$\Rightarrow x^2b^2 = a^2b^2 - a^2y^2$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ $x : 0 \to a$

$y : 0 \to \frac{b}{a}\sqrt{a^2-x^2}$.

(Ans:- $\frac{a^2b^2}{8}$)

(15) By changing the order of Integration, Evaluate

$\int_0^3 \int_1^{\sqrt{4-y}} (x+y) \, dx \, dy$.

Sol: $\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y) \, dx \, dy$.

Given limits are $x=1$ and $x = \sqrt{4-y}$ by solving

$x^2 = 4-y$.

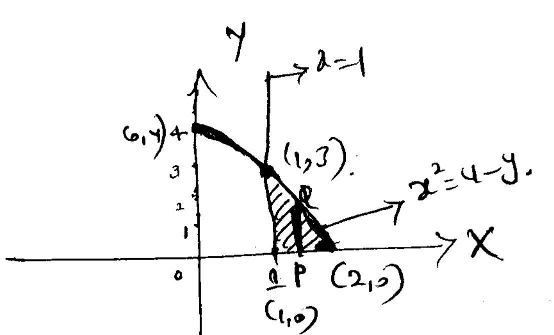
$\sqrt{4-y} = 1$

$4-y=1$

$\boxed{y=3}$

Put $\boxed{y=3} \Rightarrow \boxed{x=1}$

$\therefore (x,y) = (1,3)$



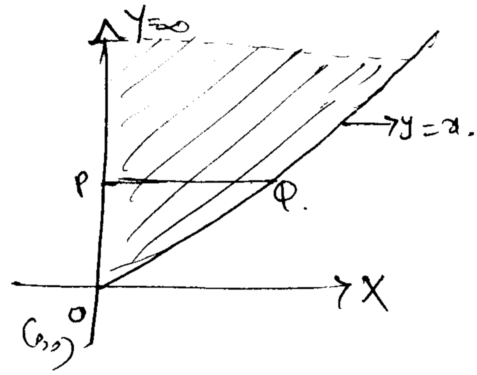
$\therefore$ $x : 1 \to 2$ & $y : 0 \to 4-x^2$

(Ans:- $\frac{241}{60}$)

(16). Evaluate the integral by changing the order of integration (16)

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$$

Sol:
$$\text{R.I} = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx.$$



For fixed 'y'.

$y : 0 \to \infty.$

$x : 0 \to y.$

$$\therefore \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \left[\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx \right] dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{y}\, x \right)_{x=0}^{y} dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{\cancel{y}} \cdot \cancel{y} \right) dy$$

$$= \left(\frac{e^{-y}}{-1} \right)_0^{\infty} = -\left(e^{-\infty} - e^{-0} \right) = -(0-1) = 1$$

(17)
$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \int_{z=0}^{\frac{a^2-r^2}{2}} r\, dz\, dr\, d\theta$$

Sol:
$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \left[\int_{z=0}^{\frac{a^2-r^2}{2}} r\, dz \right] dr\, d\theta.$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} r\, (z)_0^{\frac{a^2-r^2}{2}}\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{0}^{a\sin\theta} r\left(\frac{a^2-r^2}{2}\right) dr\, d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{0}^{a\sin\theta} (ra^2 - r^3)\, dr\, d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} \left(a^2\frac{r^2}{2} - \frac{r^4}{4}\right)_{r=0}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2}\left[\frac{a^4\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right] d\theta = \frac{a^4}{4}\int_{0}^{\pi/2} \sin^2\theta\, d\theta - \frac{a^4}{8}\int_{0}^{\pi/2} \sin^4\theta\, d\theta$$

$$= \frac{a^4}{4}\int_{0}^{\pi/2}\left[\frac{1-\cos 2\theta}{2}\right] d\theta - \frac{a^4}{8}\int_{0}^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \qquad \begin{pmatrix}\cos 2\theta = 1-2\sin^2\theta \\ \cos 2\theta = 2\cos^2\theta - 1\end{pmatrix}$$

$$= \frac{a^4}{8}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2} - \frac{a^4}{32}\int_{0}^{\pi/2}\left(1+\cos^2 2\theta - 2\cos 2\theta\right) d\theta$$

$$= \frac{a^4}{8}\left(\frac{\pi}{2}\right) - \frac{a^4}{32}\left(\theta - \frac{2\sin 2\theta}{2}\right)_0^{\pi/2} + \frac{a^4}{32}\int_{0}^{\pi/2} \cos^2 2\theta\, d\theta$$

$$= \frac{\pi a^4}{16} - \frac{a^4}{32}\left(\frac{\pi}{2}\right) - \frac{a^4}{32}\int_{0}^{\pi/2}\left(\frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{\pi a^4}{16} - \frac{\pi a^4}{64} - \frac{a^4}{64}\left(\theta + \frac{\sin 4\theta}{4}\right)_0^{\pi/2}$$

$$= \frac{\pi a^4}{16} - \frac{\pi a^4}{64} - \frac{a^4}{64}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi a^4}{16} - \frac{\pi a^4}{64} - \frac{\pi a^4}{128}$$

$$= \frac{\pi a^4}{16} - \frac{3\pi a^4}{128} = \frac{8\pi a^4 - 3\pi a^4}{128} = \frac{5\pi a^4}{128}$$

(18). $\iiint\limits_V (xy+yz+zx)\,dx\,dy\,dz$, where 'V' is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$. $(33/2)$

(19). Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

Sol:

$$I = \int_{x=0}^{\log 2}\int_{y=0}^{x}\left[\int_{z=0}^{x+\log y} e^{x+y+z}\,dz\right]dy\,dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x}\int_{z=0}^{x+\log y} e^{x}\cdot e^{y}\left(e^{z}\,dz\right)dy\,dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x}\left[e^{z}\right]_{z=0}^{x+\log y} e^{x}e^{y}\,dy\,dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x}\left(e^{x+\log y}-1\right)e^{x}e^{y}\,dy\,dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x}\left(e^{x}\cdot e^{\log y}-1\right)e^{x}e^{y}\,dy\,dx$$

$$= \int_{x=0}^{\log 2} e^{x}\left[\int_{y=0}^{x} e^{y}\left(e^{x}y-1\right)dy\right]dx$$

$$= \int_{x=0}^{\log 2} e^{x}\left[\int_{y=0}^{x}\left(ye^{x}-1\right)e^{y}\,dy\right]dx$$

$$= \int_{x=0}^{\log 2} e^{x}\left[\left(ye^{x}-1\right)e^{y}-\int e^{x}e^{y}\,dy\right]_{0}^{x}dx = \int_{x=0}^{\log 2} e^{x}\left[\left(ye^{x}-1\right)e^{y}-e^{x+y}\right]_{y=0}^{x}dx$$

$$= \int_{x=0}^{\log 2} e^{x}\left[\left(xe^{x}-1\right)e^{x}-e^{2x}+1+e^{x}\right]dx$$

$$= \int_{x=0}^{\log 2} e^{x}\left(xe^{2x}-\cancel{e^{x}}-e^{2x}+1+\cancel{e^{x}}\right)dx$$

$$= \int_{x=0}^{\log 2} e^{x}\left(xe^{2x}-e^{+2x}+1\right)dx$$

$$= \int_{x=0}^{\log 2}\left(xe^{3x}-e^{3x}+e^{x}\right)dx = \left[\frac{xe^{3x}}{3}-\int 1\cdot\frac{e^{3x}}{3}dx-\frac{e^{3x}}{3}+e^{x}\right]_{0}^{\log 2}$$

$$= \left(\frac{xe^{3x}}{3}-\frac{e^{3x}}{9}-\frac{e^{3x}}{3}+e^{x}\right)_{0}^{\log 2}$$

$$= \frac{\log 2}{3}e^{3\log 2}-\frac{e^{3\log 2}}{9}-\frac{e^{3\log 2}}{3}+e^{\log 2}+\frac{1}{9}+\frac{1}{3}-1$$

$$= \frac{8}{3}\log 2-\frac{8}{9}-\frac{8}{3}+2+\frac{1}{9}+\frac{1}{3}-1 = \frac{8}{3}\log 2-\frac{19}{9}$$

(20). Evaluate the triple Integral $\iiint xy^2z\,dx\,dy\,dz$ taken through the +ve octant of the sphere $x^2+y^2+z^2=a^2$.

Sol: Eqn of the sphere is $x^2+y^2+z^2=a^2$.

the limits of Integration are.

$z: 0 \to \sqrt{a^2-x^2-y^2}$

$y: 0 \to \sqrt{a^2-x^2}$

$x: 0 \to a.$

$$\therefore \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\left[\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z\,dz\right]dy\,dx = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy^2\left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}}dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{a} xy^2 \int_{y=0}^{\sqrt{a^2-x^2}} (a^2-x^2-y^2)\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x \int_{y=0}^{\sqrt{a^2-x^2}} \left[(a^2-x^2)y^2-y^4\right] dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x \left[\left(\frac{a^2-x^2}{3}\right)y^3-\frac{y^5}{5}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\left[\frac{(a^2-x^2)(a^2-x^2)^{3/2}}{3}-\frac{(a^2-x^2)^{5/2}}{5}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x\,(a^2-x^2)^{5/2}\left(\frac{1}{3}-\frac{1}{5}\right) dx$$

$$= \frac{1}{2}\cdot\frac{2}{15}\int_{x=0}^{a} x\,(a^2-x^2)^{5/2}\, dx$$

$$= \frac{1}{15}\int_{x=0}^{a} -\frac{dt}{2}\, t^{5/2} \qquad \left(\because\ a^2-x^2=t,\ -2x\,dx=dt\right)$$

$$= -\frac{1}{30}\left(\frac{t^{7/2}}{7/2}\right)_{a^2}^{0}$$

$$= -\frac{1}{30}\,\frac{2}{7}\left(0-a^7\right)$$

$$= \frac{a^7}{105}$$

(21). Evaluate $\iiint xyz\,dx\,dy\,dz$ over the +ve octant of the sphere $x^2+y^2+z^2=a^2$

Sol: $\left(\frac{a^6}{48}\right)$.

(22). Change of variables – (Problem)

Evaluate $\iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) dx\,dy$ over the 1st quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by using the transformation $x=au$ and $y=bv$.

Sol: Given transformations are $\left.\begin{array}{l} x=au \\ y=bv. \end{array}\right\}$ — (1)

Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow u^2+v^2=1.$ $[\because (1)]$

which represent circle.

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = ab.$$

$$\therefore \iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) dx\,dy = \iint_R \left(1-u^2-v^2\right) |J|\, du\,dv.$$

$$= \iint \left(1-u^2-v^2\right) ab\, du\,dv$$

$$= ab\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(1-r^2)\,r\,dr\,d\theta \quad \left(\because u=r\cos\theta,\ \because v=r\sin\theta\right)$$

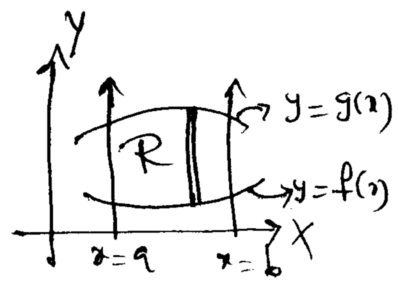
Applications :-

①. Finding the Area by using double Integral :-

Let 'R' is the region which is enclosed by $y=f(x)$, $y=g(x)$, $x=a$, $x=b$ in xy-plane. Then the area of the region 'R' is given by

$\iint_R dy\,dx$, Here

$$\iint_R dy\,dx = \int_{x=a}^{b}\int_{y=f(x)}^{y=g(x)} dy\,dx$$



If the region 'R' enclosed by $x=f(y)$, $x=g(y)$, $y=c$, $y=d$ then the Area of the region 'R' is given by

$$\iint_R dx\,dy = \int_{y=c}^{d}\int_{x=f(y)}^{x=g(y)} dx\,dy.$$

— * Problems * —

①. Find the Area enclosed by parabola $x^2=y$ — (1) & $y^2=x$ — (2).

Sol: By solving (1) & (2)

sub. Eqn(1) in Eqn(2)

$(x^2)^2 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3-1)=0 \Rightarrow x=0,\ x=1.$

Put $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1.$

the point of intersections are $O=(0,0)$

$A=(1,1)$.

let us fix 'x' variable

for fixed 'x' draw a strip $\parallel^{el}$ to y-axis and limits are

$\therefore x : 0 \to 1$

$y : x^2 \to \sqrt{x}$.

$$\text{Area} = \iint dy\,dx = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} dy \right] dx$$

$$= \int_0^1 (y)_{x^2}^{\sqrt{x}}\, dx$$

$$= 1/3.$$

(2). Find the area lying b/w the parabola $y = 4x - x^2$ — (1)
and the line $y = x$. — (2)

Sol: (1) & (2) $\Rightarrow 4x - x^2 = x \Rightarrow x^2 - 3x = 0$

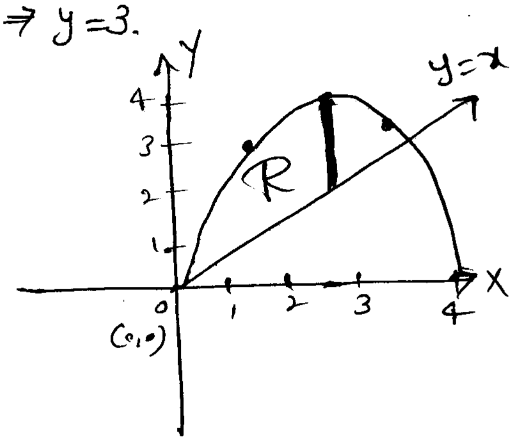
$x = 0, 3$.

Put $x = 0 \Rightarrow y = 0$

$x = 3 \Rightarrow y = 3$.

$\therefore x : 0 \to 3$

$y : x \to 4x - x^2$



$\boxed{\text{Ans :- } \frac{9}{2}.}$

$y = 4x - x^2$

x:	0	1	2	3	4
y:	0	3	4	3	0

(3). Find the Area of the region bounded by the parabolas $y^2 = 4ax$ — (1)
For fixed 'y', strip ‖el to $x^2 = 4ay$ — (2)
x-axis.

Sol:

Put $x = 0 \Rightarrow y = 0$

Put $x = 4a \Rightarrow y = 4a$.

$\therefore y : 0 \to 4a$

$x : \frac{y^2}{4a} \to 2\sqrt{ay}$

$\boxed{A :- 16a^2/3}$

S.on.b.s.

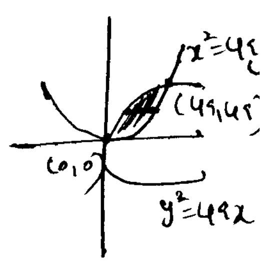
$x^4 = 16a^2y^2$

$x^4 = 16a^2(4ax)$ [$\because$ (1)]

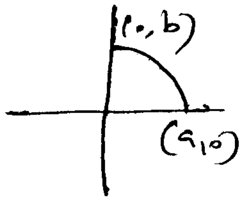
$x^4 = 64xa^3$

$\Rightarrow x^4 - 64xa^3 = 0$

$\Rightarrow x(x^3 - 64a^3) = 0 \Rightarrow x = 0, x = 4a$.



④ Find the Area of a plane in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ⑤3



Sol: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

In the 1st quadrant $y: 0 \rightarrow \frac{b}{a}\sqrt{a^2-x^2}$

subs $y=0$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\rightarrow x=\pm a$

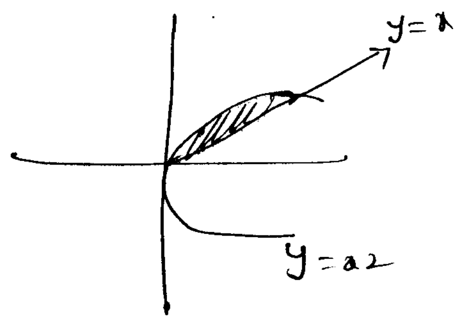
$\therefore x: 0 \rightarrow a$

$\therefore$ Area $=\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$

$=\frac{\pi ab}{4}$.

⑤. Find the Area enclosed b/w the parabola $y=x^2$ & the line $y=x$.

$\llcorner(1)$ $\llcorner(2)$



Sol: (1) & (2)

$x^2=x \Rightarrow x^2-x=0 \Rightarrow x=0, 1$

put $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$\therefore x: 0 \rightarrow 1$

$y: x^2 \rightarrow x$.

($\therefore$ 1/6).

Q. Finding the volume apply the tripe integral :-

the volume of a solid is given by

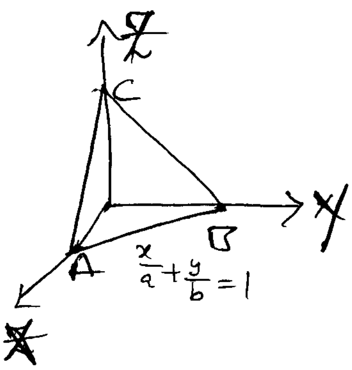
$$\iiint_V dx\,dy\,dz.$$

— * Problems * —

(1). Find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(or)

Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes by triple integral.



Sol:- $V=\iiint_V dx\,dy\,dz$.

Given $x=0$, $y=0$, $z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\Rightarrow \frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}$

$\Rightarrow z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$\therefore z: 0 \to c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Put $z=0$

$\frac{x}{a}+\frac{y}{b}=1$

$y: 0 \to b\left(1-\frac{x}{a}\right)$

Put $y=0$

$\frac{x}{a}=1 \Rightarrow x=0 \to a.$

$$V = \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} \int_{z=0}^{c(1-x/a-y/b)} dz\, dy\, dx$$

$$= \frac{abc}{6}.$$

(2) Evaluate $\iiint_V dx\, dy\, dz$, where 'V' is the finite region of the space formed by the planes $x=0, y=0, z=0$ & $2x+3y+4z=12$

Sol:
$$2x+3y+4z=12$$
$$4z = 12-2x-3y$$
$$z = \frac{12-2x-3y}{4}$$
$$\therefore z: 0 \to \frac{12-2x-3y}{4}$$
$$y: 0 \to \frac{12-2x}{3}.$$
$$x: 0 \to 6.$$
$$\therefore \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} \int_{z=0}^{\frac{12-2x-3y}{4}} dz\, dy\, dx$$

$$= 12$$

(3). Find the volume common to the cylinders $x^2+y^2=a^2$ & $x^2+z^2=a^2$

Sol: Given $x^2+y^2=a^2$ ——(1)
$x^2+z^2=a^2$ ——(2)

from (2)
$z = \pm\sqrt{a^2-x^2}$; From (1)
$y = \pm\sqrt{a^2-x^2}$

$\therefore x: -a \to a$

$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

$z: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

$$\therefore V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$\boxed{V = \frac{16a^3}{3}}$$

④. find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

(or)

find the volume of the greatest rectangular parallelopiped that

Sl^n: The solid figure $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut into 8 Equal pieces by the three coordinate planes. Hence the volume of the solid is equal to '8' times the volume of the solid bounded by $x=0$, $y=0$, $z=0$ & the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$\therefore$ $x: 0 \to a$

$y: 0 \to b\sqrt{1-\frac{x^2}{a^2}}$

$z: 0 \to c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$\therefore$ Hence the required volume $= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$

$$= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx \quad (1)$$

write $1-\frac{x^2}{a^2}=\frac{p^2}{b^2}$

$\therefore$ The required volume,

$$= 8\int_{x=0}^{a}\int_{y=0}^{p}\frac{c}{b}\sqrt{p^2-y^2}\,dy\,dx$$

$$= 8\,\frac{c}{b}\int_{0}^{a}\left[\int_{y=0}^{p}\sqrt{p^2-y^2}\,dy\right]dx \quad \text{——— (2)}$$

But $\displaystyle\int_{y=0}^{p}\sqrt{p^2-y^2}\,dy = \int_{0}^{\pi/2} p\cos\theta\cdot p\cos\theta\,d\theta$ $\quad\Big[\because$ put $y=p\sin\theta$

$dy = p\cos\theta\,d\theta$

If $y=0 \Rightarrow \theta=0$

$y=p \Rightarrow \theta=\pi/2\Big]$

$$= p^2\int_{0}^{\pi/2}\cos^2\theta\,d\theta$$

$$= p^2\,\frac{\pi}{4} = \frac{\pi}{4}\,b^2\left(1-\frac{x^2}{a^2}\right) \quad \text{——— (3)}$$

from (3) & (2)

the required volume.

$$= \frac{8c}{b}\cdot\frac{\pi}{4}b^2\int_{0}^{a}\left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_{0}^{a}$$

$$= 2\pi bc\left[a-\frac{a}{3}\right]$$

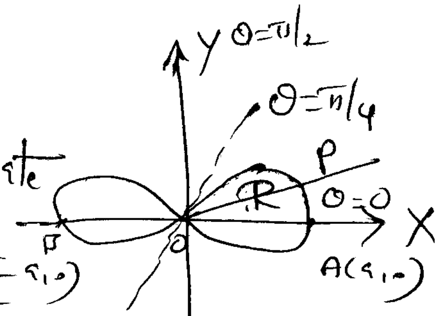
$$= 2\pi bc\left(\frac{2a}{3}\right)$$

$$= \frac{4\pi}{3}abc \text{ cubic units}$$ //

$\boxed{\text{Note}}$: putting $a=b=c$, we obtain the volume of sphere $x^2+y^2+z^2=a^2$ of $\frac{4\pi a^3}{3}$ //

(5). Find the whole area of the lemniscate $r^2 = a^2\cos 2\theta$

Sol: The curve $r^2 = a^2\cos 2\theta$ is symmetrical about both co-ordinate axes, and it passes through the pole 'O'. It intersects the Initial line at $A(a,0)$, $B(-a,0)$.



There two symmetrical loops are formed by the curve, Also each loop is symmetrical about the Initial line

∴ whole area of the lemniscate = 4 × area enclosed by one of the loops above the Initial line.

i.e $A = 4 \times \iint_R r\,dr\,d\theta$

$$A = 4 \times \int_{\theta=0}^{\pi/4} \int_{r=0}^{r=a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

[∵ 'R' drawn b/w lines $\theta = 0$ & $\theta = \pi/4$]

$$= a^2$$